Investigating Stability Analysis Issues for Fuzzy Cognitive Maps

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***Abstract*--The present paper is devoted to Fuzzy Cognitive Maps stability analysis. There are investigated the Fuzzy Bidirectional Associative Memories (FBAMs) and some common features for FCMs and FBAMs have been revealed, which may lend to use the theoretical background of FBAMS for the case of FCMs. Basing on the existing mathematical apparatus for FBAMs stability analysis the FCMs stability criterion was form and described in this research work. The proposed approach is an initiative attempt to the mathematical formulation of FCMs stability and it require further future research.**

***Index Terms*--Stability analysis, Fuzzy Cognitive Maps, FBAMS, S-T composition**

I. Introduction

Fuzzy Cognitive Maps (FCMs) is a relatively new modeling and control methodology based on the synergism of Fuzzy Logic and Neural Networks [1]. Actually, there are a lot of undisclosed questions concerning FCMs dynamics, mathematical formulation and stability examination [14].

In essence, FCM could be considered, as is a discrete nonlinear modeling methodology, where known methods for discrete and nonlinear systems stability analysis could be used. But on the other hand, there is great difficulty for the mathematical formalization of FCMs along with the conventional methods for analysis.

Here it is followed an analysis approach taking advantage on the fact that FCMs combine Neural Networks and Fuzzy logic and on existing mathematical operations that accepted and used within these theories. Kosko examined Associative Memories stability by identifying a Lyapunov or energy function with an associative memory states [2][3]. Further it is proved that a Bidirectional Associative Memory is stable for some values of the network's weights. Kosko [4] proves that Lyapunov techniques yield sufficient conditions for stability for the simplest feedback SAM (Standard Additive Model) system. FCMs extend the additive structure to the dynamical systems where each rule feeds all other rules. Their dynamics can range from those of feedback neural networks to more complex semantic networks of embedded SAMs or other systems. Such systems demand stronger conditions for stability analysis.

Similar approaches has been based on state-dependent Lyapunov function to reduce the conservatism in stability analysis of Takagi-Sugeno (T-S) fuzzy systems via Linear Matrix Inequalities (LMIs) [5], where there are defined the conditions when a T-S system is stable or not. Other approaches are based on LMIs for the robust stability synthesis [6]. More classical approaches are based on the description of a system using unstable differential equations and solving them using minimization of a combined integral criterion or Volter kernels formalism [7] and other methods are used for digital systems analysis are shown [8].

The stability problem for Fuzzy Bidirectional Associative Memories (FBAMs) is investigated in [9]. There is used a conventional mathematical apparatus oriented on fuzzy and neural systems and propose several criteria for stability analysis. This work has intricate the present research and it is considered quite close.

It the following of the paper, it will be shown that FCMs and FBAMs have several common features, which allow applying similar stability analysis methods to both with different reservations and restrictions. Actually, FCM is considered here as a special case of FBAM.

II. The Mathematical Representation and Main Features of FBAMs

FBAM in neural network interpretation is a two-layer hierarchy of connected fuzzy neurons called layer X and layer Y respectively (Fig. 1).

$$P = \left(p_{ij}\right)_{n\times m},\ p_{ij} \in [0;1]$$

is the connection matrix from the layer X to the layer Y and

$$R = \left(R_{ji}\right)_{m\times n},\ r_{ji} \in [0;1]$$

is a connection matrix from the layer Y to the layer X.

Here n and m are numbers of fuzzy neurons in the layers X and Y respectively.

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The iterative conversion of FBAM states $\left(X^{(k)}, Y^{(k)}\right)$ is presented by the following formula:

$$X^{(k+1)} = \left(X^{(k)} \circ P\right) \circ R,$$

$$Y^{(k+1)} = \left(Y^{(k)} \circ R\right) \circ P.$$

Here $\circ$ is a matrix product in the sense of S-T composition.

The FBAMs mathematical representation is given in more details in [9].

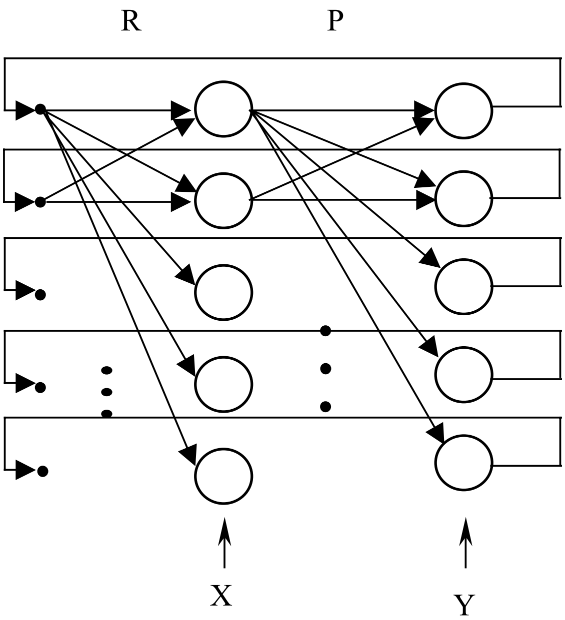


Figure 1. FBAMs graphical representation.

III. The Fuzzy cognitive map structure

FCM is represented as a signed graph permitting feedback where concepts are interconnected with weighted arcs, which represent causal relationship between nodes [12] (Fig. 2). The graphical structure of FCM is a one-layer network with weighted interconnections, which can be modified using training algorithms such as Hebbian learning [3].

The FCM structure is characterized by the value of each concept and the value of each weight connecting concepts [13].

- Every concept has a value A_i, which describe the corresponding system variable and may be is the transformation of the physical measurement or the calculated value in the FCM. The value A_i should belong to the interval $[0;1]$.
- The Weight W_{ij} of each interconnection shows: to which degree the i-th concept influences on the j-th. Weight W_{ij} range in the interval $[-1;1]$. There exist three relationships between concepts: positive ($W_{ij} > 0$), negative ($W_{ij} < 0$) or no relationship ($W_{ij} = 0$).

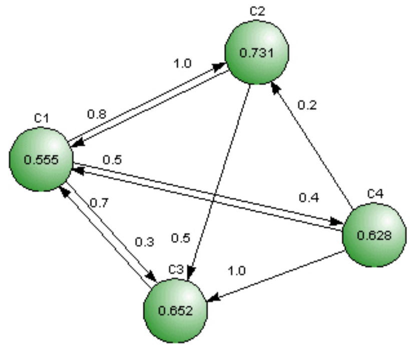


Figure 2. A very simple FCM graphical representation.

In the framework of FCM, the value A_i of concept i at the simulation time step t is calculated using the following equation:

$$A_i^t = f\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} A_j^{t-1} W_{ij} + W_{ii} A_i^{t-1}\right),$$

where A_i^t – the value of concept C_i at time t;

A_i^{t-1} – the value of concept C_i at time $t-1$;

A_j^{t-1} – the value of concept C_j at time $t-1$;

W_{ij} – the causal weight from concept C_i to concept C_j;

W_{ij} – represents the degree with which the previous value of each concept is involved in the calculation of next value;

f – the threshold function which squashes the summation result in the interval $[0;1]$

IV. T-norms and FBAMs stability

Cheng and Fan [9] justified their theory for FBAMs stability analysis using triangular norms that will be discussed here.

Triangular norms (T-norms) T generalize the AND operation for fuzzy logic and T-conorm S is an analog of OR operation. T-norm and T-conorm satisfy the axioms which show that it is an associative, commutative, non-decreasing binary operation on [0;1].

These axioms are the following:

t-norm (T)	t-conorm (S)
$T(0,0)=0$	$S(1,1)=1$
$T(x,1)=T(1,x)=x$	$S(x,0)=S(0,x)=x$
$T(x,y)=T(y,x)$	$S(x,y)=S(y,x)$
$T(u,v)\le T(x,y)$: $u\le x, v\le y$	$S(u,v)\le S(x,y)$: $u\le x, v\le y$
$T(T(x,y),z)=T(x,T(y,z))$	$S(S(x,y),z)=S(x,S(y,z))$

T-norms and T-conorms have the following feature:

$T(x,y)\le \min(x,y)\le \max(x,y)\le S(x,y)$.

The product of two matrices in the sense of S-T composition is defined as follows:

If there are given fuzzy matrices $A=(a_{ij})_{m\times p}$ and $B=(b_{ij})_{p\times n}$ then the S-T composition is defined as follows:

$A\circ B=(c_{ij})$,

where

$$c_{ij}=(a_{i1}Tb_{1j})S(a_{i2}Tb_{2j})S...S(a_{ip}Tb_{pj}). \quad (1)$$

The following have proved the FBAM stability and it is appropriate to show them here for the further reasoning being more easily understood.

Proposition 1 *[9]: Let S and T be both T-conorms or T-norms. Then any FBAM based on the S-T composition is globally stable.*

Theorem 1 *[9]: Let T be a T-norm. Let W and R be the connection matrices of the FBAM based on the max-T composition. Then the FBAM is globally stable if and only if $(W\circ R)$ is convergent in the sense of max-T composition.*

This theorem proof shows that global stability study for FBAMs lies on matrix power convergence.

V. FCMs REPRESENTATION THROUGH FBAM STRUCTURE

The main difference between FCMs and FBAMs is their representation. FCM is a one-layer network whereas FBAM is a two-layer hierarchy. In particular, two-layer structure allows speaking of S-T composition of weight matrices.

FCM, in contrast to FBAM, has got only one layer and also only one weight matrix. Furthermore, this weight matrix has not any triangular superdiagonal form, in comparison to the two weight matrices for FBAM. Consequently, on the initial step we cannot examine the mentioned above method for stability analysis to FCMs. Indeed, if we can transfer FCM to such a form we can do it. In other words, it is necessary to decompose the weight matrix W so that it would be a multiplication result of two triangular matrices:

$W=L\cdot U$.

There are a lot of classical algorithms for this problem solution. One appropriate method could be Cholesky factorization, which allows factorize a matrix into two superdiagonal matrices R and R^T (transpose of R):

$W=R\cdot R^T$

But this algorithms holds if and only if matrix W is positive-definite and symmetric. But the latter condition is rarely fulfilled in FCMs. It demands that every concept C_i influences on a concept Cj to the same degree as a concept Cj influences on a concept Ci, i.e. $w_{ij}=w_{ji}$ for every i=1,…,n, j=1,…,n. This can be distinguished very rarely according to destination of FCMs to model complex polyvalent systems with intricate interconnections.

Another approach for decomposing the FCM weight matrix is LU factorization, or Gaussian elimination, which expresses any square matrix W as the product of a permutation of a lower triangular matrix and an upper triangular matrix:

$W=L\cdot U$.

where L is a permutation of a lower triangular matrix with ones on its diagonal and U is an upper triangular matrix.

For being factorized into two matrices L and U the weight matrix W for the given FCM must satisfy the following theorem.

Theorem 2. *[9] If all the principal minors of the quadratic matrix W are non-zero then there exist upper triangular matrix U and lower triangular matrix L such that $W=L\cdot U$. If diagonal elements of L or U are non-zero then such a factorization is unique for the given W.*

The last theorem holds for every non-fuzzy matrices. But the weight matrix W for an FCM is fuzzy, so its elements are not numbers but characterize some intervals described by linguistic variables like “positive very small”, “positive small”, “positive medium” etc. It means that a fuzzy matrix always has non-zero elements and, consequently, non-zero principal minors, so it can always be factorized into two triangular matrices. The physical meaning of this position in the sense of an FCM construction is that never can one say that C_i do not influence on C_j if the given FCM is a unified interconnected structure. Even if there is no direct connection between two concepts they can influence each other through other concepts they are connected to.

So, an FCM can be transformed to a two-layer (layer G and layer H, see Fig. 3) hierarchy where weight matrix between layer G and layer H is L and weight between layers H and G is U. Number of nodes in every layer is equal to number of the FCM nodes.

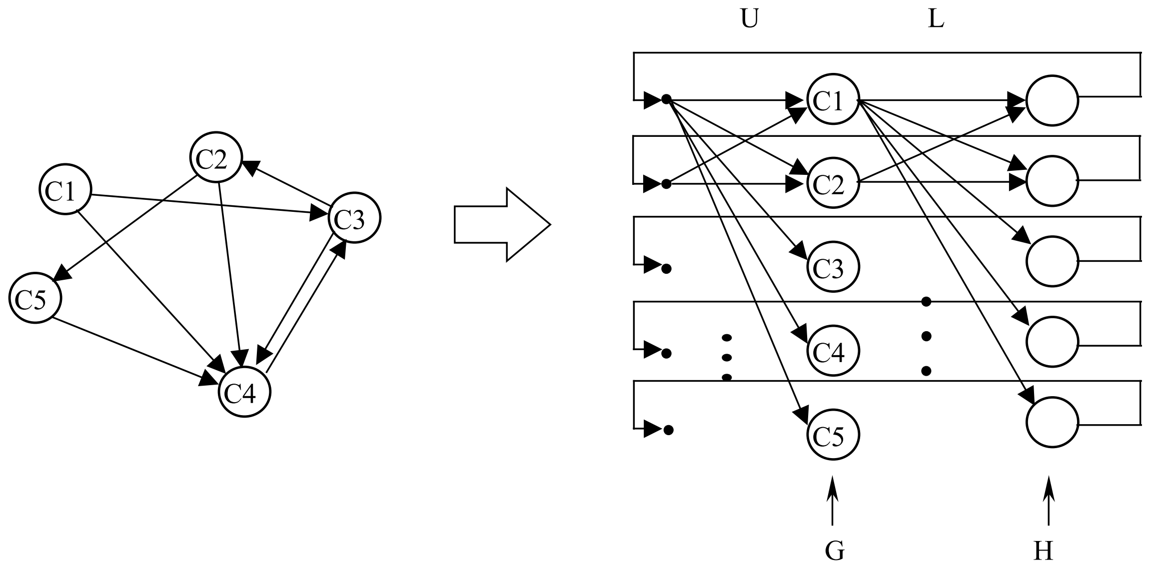


Figure 3. FCM transformation as a two-layer hierarchy.

Table 1. The most useful T-norms and T-conorms

Name T-norm	T-norm	T-conorm
Product	$T(x,y) = xy$	$S(x,y) = x + y - xy$
Dubois-Prade	$D_p(x,y) = \dfrac{xy}{\max(p,x,y)}$, $0 \le p \le 1$	$P_p(x,y) = \dfrac{x + y - xy - \min(1-p,x,y)}{\max(p,1-x,1-y)}$, $0 \le p \le 1$
Yager	$Y_p(x,y) = 1 - \min\left(1, \left[(1-x)^p + (1-\right.\right.$ $p > 0$	$Z_p = \min\left(1, \left[x^p, y^p\right]^{\frac{1}{p}}\right)$, $p > 0$

Then recursion equations for FCM can be rewritten as follows:

$$G^{(k+1)} = \left(G^{(k)} \cdot L\right) \cdot U,$$

$$H^{(k+1)} = \left(H^{(k)} \cdot U\right) \cdot L.$$

Here the operation of matrix product is used.

We apply the FCM weight matrix factorization in two matrices, one of which is upper triangular and another is lower triangular. Meantime both weight matrices for FBAM are upper triangle matrices with zero leading diagonal. However the mentioned above theorems for FBAMs are proved without commonality restriction for any fuzzy matrices. So, as a direct consequence of the mentioned above theorems, the following proposition can be posed.

Proposition 2. *Let an FCM with weight matrix W is given, and W is factorized into two triangular matrices L and U. The FCM is globally stable if its weight matrix W is based on S-T composition of matrices L, U.*

Proof of the present Proposition follows directly from the one for FBAMs [9] and is based on the properties of T-norms and T-conorms mentioned above.

Suppose that $L = (l_{ij})_{n \times n}$ and $U = (u_{ji})_{n \times n}$ are the weight matrices of an FCM received from LU factorization of the weight matrix W and $(G^{(k)}, H^{(k)})$, $k \ge 1$ is a sequence of reverberation states of FCM activated by $(G^{(0)}, H^{(0)})$. The demand for the FCM is stable will be the proposition that $(G^{(k)}, H^{(k)})$ is convergent.

If T and S are both the T-norms the T-norms' properties give the following:

$$g_i^{(k)} \le g_i^{(k+1)} \le 1,$$

$$h_j^{(k)} \le h_j^{(k+1)} \le 1$$

$$\exists k \ge 1,\ \ 1 \le i \le n,\ \ 1 \le j \le n.$$

Similarly, if T and S are both the T-conorms the following inequalities yield:

$$g_i^{(k)} \ge g_i^{(k+1)} \ge 0,$$

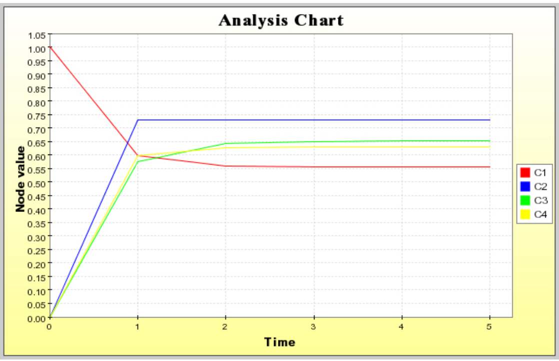
$$h_j^{(k)} \ge h_j^{(k+1)} \ge 0$$

$$\exists k \ge 1,\ \ 1 \le i \le n,\ \ 1 \le j \le n.$$

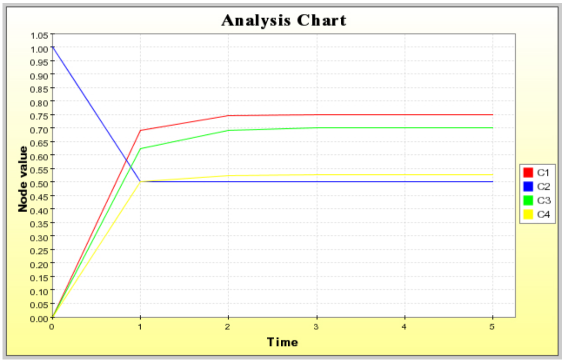
Consequently, $(G^{(k)}, H^{(k)})$ is convergent. ■

There are a lot of types of T-norms and T-conorms (many of them are listed in [4]) and all of them give absolutely different result for matrix composition – it is enough to look at their representation at table 1.

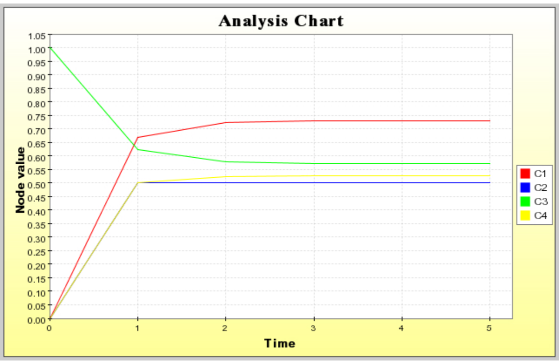
Hence, the problem of stability analysis for FCMs cannot be considered just as a task of an arbitrary S-T composition computation. This question should be posed differently: whether there exist a T-norm such that S-T composition of the triangular matrices received from the LU factorization of the weight matrix W is equal to the original matrix. If it exists then the FCM is stable. The latest will be shown with help of the simple example.



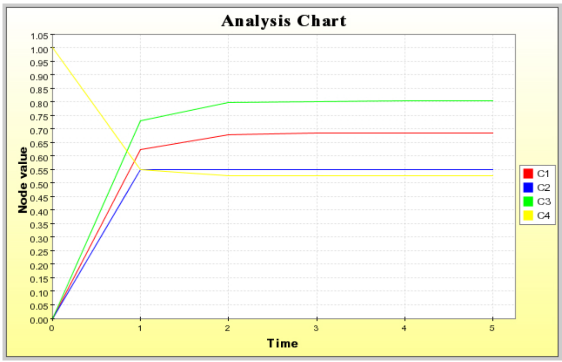
a. $\overline{A}^1 = [1 \quad 0 \quad 0 \quad 0]$



b. $\overline{A}^2 = [0 \quad 1 \quad 0 \quad 0]$



c. $\overline{A}^3 = [0 \quad 0 \quad 1 \quad 0]$



d. $\overline{A}^4 = [0 \quad 0 \quad 0 \quad 1]$.

Figure 4. Simulation results for FCM.

Example 1. *It is given an FCM (Fig. 2) with weight matrix:*

$$W = \begin{bmatrix} 0.4 & 1.0 & 0.3 & 0.4 \\ 0.8 & 0.0 & 0.5 & 0.0 \\ 0.7 & 0.0 & 0.5 & 0.0 \\ 0.5 & 0.2 & 1.0 & 0.2 \end{bmatrix}.$$

Let us check if this system is stable.

The weight matrix W is factorized to matrices L and U as follows:

$$L = \begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.875 & 0.0 & 0.0923 & 0.0 \\ 0.625 & 0.2 & 1.0 & 0.0 \end{bmatrix},$$

$$U=\begin{bmatrix}0.8 & 0.0 & 0.5 & 0.0\\ 0.0 & 1.0 & 0.05 & 0.4\\ 0.0 & 0.0 & 0.6775 & 0.12\\ 0.0 & 0.0 & 0.0 & -0.0111\end{bmatrix}.$$

If we use a product S-T composition (see Table 1) the result is the following weight matrix

$$W'=\begin{bmatrix}0.4 & 1.0 & 0.2875 & 0.4\\ 0.8 & 0.0 & 0.5 & 0.0\\ 0.7 & 0.0 & 0.4726 & 0.0001\\ 0.5 & 0.2 & 0.9278 & 0.1904\end{bmatrix}.$$

As one can see, the received matrix W' is nearly equal to the original matrix W; in the sense of fuzzy logic some deviations do not matter if they lay to the range defined by the membership function for some linguistic variables.

This result is confirmed by the simulation, which has been hold for the given FCM (fig. 2) for four input vectors:

$$\overline{A}^1=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix};$$

$$\overline{A}^2=\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix};$$

$$\overline{A}^3=\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix};$$

$$\overline{A}^4=\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}.$$

The simulation result is shown on Fig. 4 (a, b, c, d). As one can see, the process is really stable.

Thus, stability analysis for FCMs can be fulfilled according to the following simple algorithm:

1. Weight matrix factorization (for example, with Gaussian elimination): $W=L\cdot U$;

2. Receiving a product in sense of S-T composition for matrices L and U (according to Equation 1);

3. If there exist such an S-T composition of matrices L and U that the result of this composition is equal to W then the given FCM is globally stable. If one cannot find such a composition or if its result is not equal to the original W then the proposed method doesn't solve the stability problem for the given FCM.

VI. CONCLUSION

Here it was discussed some issues on FCMs stability analysis, our purpose is not to present a whole theory for stability analysis of FCM but to present issues and challenges for further investigation. The research work is proposing the examination of FCM stability analysis using the fact that there are some common features for FCMs and FBAMs. Taking advantage of the existing mathematical apparatus for FBAMs stability analysis an approach that may lend to the FCMs stability criterion was presented. There was described a new approach for FCM stability analysis that was implemented for a simple example.

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